EAST-WEST
Journal of **Economics and Business**
Vol. XXV – 2022, No 2

A SHORT NOTE ON ALGORITHMIC APPROACHES TO IMPLIED VOLATILITY OF AN OPTION

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Abstract

This paper is concerned with the implied volatilities of financial derivatives, specially options. While analyzing the option including its price, or implied volatility, some algorithms are needed which a unified version of algorithm is proposed. First, some recursive implied volatilities are derived using the Monte Carlo-based Newton-Raphson (NR) and the stochastic approximation (SA) method. Then, to extend the work of Liao (2003) in call option derivatives to almost all financial derivatives, a Bayes filter is fitted to implied volatility. Then, the existence of arbitrage opportunity in an option market is surveyed using optimal. Finally, a conclusion section is also given.

Keywords: Arbitrage opportunity, Implied volatility, NR approximation, Optimal control, SA method

JEL Classification: G12

Introduction

To use the Black-Scholes (BS) formula for option pricing, it is necessary to know the standard deviation (volatility) of underlying asset, say stock. As well as, to correspond the BS price of an option to its market price, for avoiding the arbitrage opportunity, the implied volatility, instead of actual volatility, is used in BS formula. There are many methods for deriving the implied volatility in option pricing problem, say the NR approach, see Xing (2014). Let $C(v)$ be the price of a call option with strike price K, maturity time T, risk-free rate r, and volatility v, when the price of for example stock is S. Indeed,

$$C(v) = SN(d_1) - Ke^{-rT}N(d_2),\ d_1 = \frac{\log\left(\frac{S}{K}\right) + \left(r - \frac{v^2}{2}\right)T}{v\sqrt{T}},$$
$$d_1 - d_2 = v\sqrt{T}$$

and N is the standard normal cumulative distribution function. Xing (2014) showed that

$$C'(v) = S\sqrt{T}N'(d_1).$$

Then, for each time t, the recursive volatilities are given by $v_{i+1} = v_i - \frac{C(v_i)}{C'(v_i)}, i = 1,2,\ldots$ until convergence. As follows, three recursive relations are proposed. The main contribution of the current paper, is to propose these relations for estimating the implied volatility.

(a) NR formulation. Notice that

$$C_t(v) = e^{-r(T-t)}E_v^Q(\max(S_T - K, 0)\,|\mathrm{F}_t),$$

is the call option price at time t, Q is the risk-neutral probability measure equivalent to actual probability measure and F_t is the σ-field generated by $S_u, u \leq t$. Since $C(v)$ is an expectation, it is interested to estimate it using a Monte Carlo method. Although, $C(v)$ and $C'(v)$ have closed forms in the case of call option (or put option) assuming the dynamic of S is defined by a stochastic differential equation process influenced by the Brownian motion as a random noise, however, these assumptions aren't always correct. For example, when S obey a jump process or the implied volatility of a complicated financial derivative (say, exotic option) is interested, the need of approximating the price of financial derivative or its partial differential with respect to volatility parameter using a Monte Carlo method is felt.

Gelman (1995) proposed the Monte Carlo-based method of moment approach. Here, his work is reviewed, briefly. Suppose that X is a random variable with density function $p_\theta(\cdot)$, indexed by unknown parameter θ and let

$$\mu(\theta) = E_\theta(h(X)) < \infty,$$

for some measurable function h. Assuming, $X_j, j = 1, \dots, n$ is an *iid* sample copies of X, then

$$\hat{\mu}(\theta) = \frac{\sum_{j=1}^{n} h(X_j)}{n}$$

and

$$\hat{\mu}'(\theta) = \frac{1}{n}\sum_{j=1}^{n} h(X_j)\frac{\partial}{\partial\theta}\log\left(p_\theta(X_j)\right).$$

To solve, $\mu(\theta) = \mu_0$, with respect to θ, for some pre-specified value μ_0, then the NR repetitions are given by

$$\theta_{i+1} = \theta_i - \frac{\hat{\mu}(\theta_i)-\mu_0}{\hat{\mu}'(\theta_i)}, i = 1,2, \dots,$$

again until convergence. The main idea behind the Gelman's (1995) work is the importance sampling. Thus, using this technique and this fact that $C_t(v) = c_0$, therefore,

$$v_{it+1} = v_{it} - \frac{C_t(v_{it}) - c_0}{c_t'(v_{it})}.$$

(b) SA formulation. The SA technique is a generalization of NR method (see, Robbins and Monro, 1951). Here, the SA formulation of above problem is proposed. For comprehensive review in SA, see Borkar (2008). To this end, let a_i's be a sequence of constants such that $\sum_{i=1}^{\infty} a_i = \infty$ and $\sum_{i=1}^{\infty} a_i^2 < \infty$. Then, θ_i's based on stochastic approximation are defined as follows:

$$\theta_{i+1} = \theta_i + a_i(\mu(\theta_i) - \mu_0).$$

Thus,

$$v_{it+1} = v_{it} - a_i(C_t(v_{it}) - c_0).$$

Remark 1. Stochastic volatility models say ARCH and GARCH models plays important rules in financial time series. When these models are applied to observed prices of option, a recursive relation for variance of option (a filter, i.e., GARCH filter) is constructed. However, since the option is a function of stock, say $h(s_t)$, according to the delta-method the variance of option is $(h'(s_t))^2 var(s_t)$. By dividing GARCH equation of variance of option over $(h'(s_t))^2$, a new GARCH filter is obtained for implied volatility of s_t.

Bayes filtering

Following Liao (2003), there are two main approaches to compute v. The first method suggests the use the historical data to fit a model to v, say, EWMA, GARCH and etc., to forecast v in future times. The second method is the implied volatility which uses the Black and Scholes formula and the market price of derivative m to compute v. In this sense, the volatility surface is useful tool. Liao (2003) applied the Bayes filter to utilize both approach. Here, following Liao (2003), the observation equation is given by

$$m_n = f_n + \varepsilon_n = f(v_n) + \varepsilon_n = BS(v_n) + \varepsilon_n, n \geq 1,$$

where m, f are the market price and the BS price of derivative security, respectively. The BS price is the price derived of BS pde equation where is discredited numerically by a suitable method like Euler tool in discrete time points $n \geq 1$. Next, consider the NR recursive of implied volatility

$$v_{n+1} = v_n - \frac{C_t(v_n) - c_0}{c_t'(v_n)}.$$

Now, consider the first equation as a measurement equation and the second equation as a state equation. Applying the Liao (2003) work in this section, the implied volatility is estimated using Kalman filtering. This approach is applicable when there are some noises in the observed price of option.

Arbitrage detection

Arbitrage is the fault of an economy and the no arbitrage assumption is a fundamental assumption for financial derivatives specially options. Here, using the optimal control and Bayesian mixture time series approaches, the arbitrage opportunities in option market are detected. Based on no arbitrage arguments, BS derived a pde for pricing a derivative security f defined on underlying asset S (say stock) (see, Pascucci, 2011). It is given as follows

$$\frac{\partial f}{\partial t} + rs\frac{\partial f}{\partial s} + \frac{vs^2}{2}\frac{\partial^2 f}{\partial s^2} = rf,$$

where r is the risk free rate and v is the volatility of return of S. The estimation of parameter v is too important problem to solve the above pde. Next, for simplicity arguments, suppose that $\frac{\partial f}{\partial t} = 0$. Notice that

$$m_t = f_t + \varepsilon_t.$$

To control ε_t's, it is enough to

$$min_f \int (m_t - f_t)^2 dt$$

Where

$$\frac{vs^2}{2} f'' + rsf' = rf.$$

Let $g = f'$. Then,

$$\frac{vs^2}{2} g' + rsg = rf.$$

Thus, $f' = g$ and

$$g' = \frac{2r(f - sg)}{vs^2}.$$

This defines an optimal control problem. The maximum principle implies that

$$H = (f - m)^2 - p_1 g - p_2 \frac{2r(f - sg)}{vs^2}.$$

Notice that $\frac{\partial H}{\partial s} = 0$. Thus, $f = 0.5sg$. Also,

$$\frac{\partial H}{\partial f} = 2(f - m) - p_2 \frac{2r}{vs^2} = p_1'$$
$$\frac{\partial H}{\partial g} = -p_1 + \frac{2r}{vs} p_2 = -p_2'.$$

Solving the system of equations

$$\begin{cases} -2(f - m) + p_2 \dfrac{2r}{vs^2} = p_1' \\ p_1 - \dfrac{2r}{vs} p_2 = p_2' \end{cases}$$

and $f = 0.5sg$ is seen that $\frac{f'}{f} = \frac{2}{s}$. Thus, $f = s^2$. The above solution gives the value of implied volatility.

Remark 2. Bayesian mixture. Asako and Liu (2013) used a Bayesian mixture model for detect the speculative bubbles. Indeed, the bubble price is the difference of ideal and actual of investment commodity. A similar situation occurs in the arbitrage opportunities exists in option market. Yao (1984) used the similar procedure to detect the change points and Habibi *et al*. (2017) used the Yao's (1984) approach to detect the faults in a Markov jump system. Asako and Liu (2013) modeled the bubbles prices with the time varying of an AR(1). The probability of being live of a bubble during the time vanishes. The supply and demand mechanism removes the arbitrage assumption.

Conclusions

An important parameter in option pricing is the implied volatility. Two main recursive relations for implied volatility are obtained, here, are NR and SA relations. When, there are some noises in the observed prices of option, and then the Bayes filter is used to compute the implied volatility. Optimal control is used to check the no arbitrage assumption which is fundamental assumption for option pricing and to check possible opportunity for profit.

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