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ON A DUOPOLY GAME WITH HOMOGENEOUS PLAYERS AND A QUADRATIC DEMAND FUNCTION

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ABSTRACT

In this study we investigate the dynamics of a nonlinear discrete-time duopoly game, where the players have homogeneous expectations. We suppose that the demand is a quadratic function and the cost function is linear. The game is modeled with a system of two difference equations. Existence and stability of equilibria of this system are studied. We show that the model gives more complex chaotic and unpredictable trajectories as a consequence of change in the speed of adjustment of the players. If this parameter is varied, the stability of Nash equilibrium is lost through period doubling bifurcations. The chaotic features are justified numerically via computing Lyapunov numbers and sensitive dependence on initial conditions.

Keywords: Cournot duopoly game; Discrete dynamical system; Homogeneous expectations; Stability; Chaotic Behavior.

JEL Classification: C62, C72, D43.

Introduction

An Oligopoly is a market structure between monopoly and perfect competition, where there are only a few number of firms in the market producing homogeneous products. The dynamic of an oligopoly game is more complex because firms must consider not only the behaviors of the consumers, but also the reactions of the competitors i.e. they form expectations concerning how their rivals will act. Cournot, in 1838 has introduced the first formal theory of oligopoly. He treated the case with naive expectations, so that in every step each player (firm) assumes the last values that were taken by the competitors without estimation of their future reactions.

Expectations play an important role in modelling economic phenomena. A producer can choose his expectations rules of many available techniques to adjust his production outputs. In this paper we study the dynamics of a duopoly model where each firm behaves with homogeneous expectations strategies. We consider a duopoly model where each player forms a strategy in order to compute his expected output. Each player adjusts his outputs towards the profit maximizing amount as target by using his expectations rule. Some authors considered duopolies with homogeneous expectations and found a variety of complex dynamics in their games, such as appearance of strange attractors (Agiza, 1999, Agiza et al., 2002, Agliari et al., 2005, 2006, Bischi, Kopel, 2001, Kopel, 1996, Puu,1998). Also models with heterogeneous agents were studied (Agiza, Elsadany , 2003, 2004, Agiza et al., 2002, Den Haan , 20013, Fanti, Gori, 2012, Tramontana , 2010, Zhang , 2007).

In the real market producers do not know the entire demand function, though it is possible that they have a perfect knowledge of technology, represented by the cost function. Hence, it is more likely that firms employ some local estimate of the demand. This issue has been previously analyzed by Baumol and Quandt, 1964, Puu 1995, Naimzada and Ricchiuti, 2008, Askar, 2013, Askar, 2014. Efforts have been made to model bounded rationality to different economic areas: oligopoly games (Agiza, Elsadany, 2003, Bischi *et al*, 2007); financial markets (Hommes, 2006); macroeconomic models such as multiplier-accelerator framework (Westerhoff,2006). In particular, difference equations have been employed extensively to represent these economic phenomenons (Elaydi, 2005; Sedaghat, 2003). Bounded rational players (firms) update their production strategies based on discrete time periods and by using a local estimate of the marginal profit. With such local adjustment mechanism, the players are not requested to have a complete knowledge of the demand and the cost functions (Agiza, Elsadany, 2004, Naimzada, Sbragia, 2006, Zhang *et al*, 2007, Askar, 2014). All they need to know is if the market responses to small production changes by an estimate of the marginal profit. The paper is organized as follows: In Section 2, the dynamics of the duopoly game with homogeneous expectations, quadratic demand and linear cost functions is analyzed. The existence and local stability of the equilibrium points are also analyzed. In Section 3 numerical simulations are used to show complex dynamics via computing Lyapunov numbers, and sensitive dependence on initial conditions.

The game

In oligopoly game players can choose simple expectation rules such as naïve or complicated as adaptive expectations and bounded rationality. The players can use the same strategy (homogeneous expectations) or can use different strategy (heterogeneous expectations). In this study we consider two boundedly rational players such that each player think with the same strategy to maximize his output. We consider a simple Cournot-type duopoly market where firms (players) produce homogeneous goods which are perfect substitutes and offer them at discrete-time periods $t = 0,1,2,...$ on a common market. At each period t, every firm must form an expectation of the rival's output in the next time period in order to determine the corresponding profit-maximizing quantities for period $t+1$. The inverse demand function of the duopoly market is assumed quadratic and decreasing:

$$P = a - b(x+y)^2 \tag{1}$$

and the cost functions are:

$$C_1(x) = cx,\ C_2(y) = cy \tag{2}$$

where $Q = x + y$ is the industry output and $a,b,c > 0$. With these assumptions the profits of the local firms are given by

$$\begin{aligned} \Pi_1(x,y) &= x[a - b(x+y)^2] - cx \\ \Pi_2(x,y) &= y[a - b(x+y)^2] - cy \end{aligned} \tag{3}$$

Then the marginal profit of the firm at the point (x, y) of the strategy space is given by

$$\begin{aligned} \frac{\partial \Pi_1}{\partial x} &= a - c - b(x+y)^2 - 2b(x+y)x \\ \frac{\partial \Pi_2}{\partial y} &= a - c - b(x+y)^2 - 2b(x+y)y \end{aligned} \tag{4}$$

We suppose that each first firm decides to increase its level of adaptation if it has a positive marginal profit, or decreases its level if the marginal profit is negative (bounded rational player). If k > 0 the dynamical equations of the players are:

$$x(t+1) - x(t) = k\frac{\partial \Pi_1}{\partial x},\ \mathrm{y}(t+1) - y(t) = k\frac{\partial \Pi_2}{\partial y} \tag{5}$$

The dynamical system of the players is described by

$$\begin{cases} x(t+1) = x(t) + k[a - c - b(x+y)^2 - 2b(x+y)x] \\ y(t+1) = y(t) + k[a - c - b(x+y)^2 - 2b(x+y)y] \end{cases} \tag{6}$$

We will focus on the dynamics of the system (6) to the parameter k

The equilibria of the game

The equilibria of the dynamical system (6) are obtained as nonnegative solutions of the algebraic system

$$\begin{cases} a - c - b(x+y)^2 - 2b(x+y)x = 0 \\ a - c - b(x+y)^2 - 2b(x+y)y = 0 \end{cases} \tag{7}$$

which obtained by setting $x(t+1) = x(t),\ \mathrm{y}(t+1) = y(t)$ in Eq. (6) and we can have one equilibrium $E^* = (x^*, y^*)$, where

$$x^* = \mathrm{y}^* = \left(\frac{a-c}{8b}\right)^{\frac{1}{2}} \tag{8}$$

The equilibrium E^* is called Nash equilibrium, provided that $a > c$. The study of the local stability of equilibrium solution is based on the localization on the complex plane of the eigenvalues of the Jacobian matrix of the two dimensional map (Eq. (9)).

$$\begin{cases} f(x,y) = x + k[a - c - b(x+y)^2 - 2b(x+y)x] \\ g(x,y) = y + k[a - c - b(x+y)^2 - 2b(x+y)y] \end{cases} \tag{9}$$

In order study the local stability of equilibrium points of the model Eq.(6), we consider the Jacobian matrix along the variable strategy (x, y)

$$J(x,y)=\begin{bmatrix} 1-2bk(3x+2y) & -2bk(2x+y) \\ -2bk(x+2y) & 1-2bk(2x+3y) \end{bmatrix} \tag{10}$$

The Nash equilibrium E^* is locally stable if the following conditions are hold

$$\begin{cases} 1-T+D>0 \\ 1+T+D>0 \\ 1-D>0 \end{cases} \tag{11}$$

where $T = 2-20kbx^*$ is the trace and $D = 64(kbx^*)^2 - 20kbx^* + 1$ is the determinant of the Jacobian matrix

$$J(E^*)=\begin{bmatrix} 1-10kbx^* & -6kbx^* \\ -6kbx^* & 1-10kbx^* \end{bmatrix} \tag{12}$$

The first condition

$$1-T+D>0 \Leftrightarrow 64(kbx^*)^2>0 \tag{13}$$

is always satisfied

The second and third conditions are the conditions for the local stability of Nash equilibrium which becomes:

$$\begin{cases} 1+T+D>0 \\ 1-D>0 \end{cases} \Leftrightarrow \begin{cases} 64(kbx^*)^2-40kbx^*+4>0 \\ kbx^*(20-64kbx^*)>0 \end{cases} \tag{14}$$

From Eq.(14) it follows that the Nash equilibrium is locally stable if

$$0 < kbx^* < 0.125 \Leftrightarrow 0 < k < \frac{0.125}{bx^*} \tag{15}$$

Numerical Simulations

To provide some numerical evidence for the chaotic behavior of the system Eq. (6), as a consequence of change in the parameters k , we present various numerical results here to show the chaoticity, including its bifurcations diagrams, Lyapunov numbers and sensitive dependence on initial conditions (Kulenovic, M., Merino, O., 2002). In order to study the local stability properties of the equilibrium points, it is convenient to take $a = 12, b = 1, c = 2$. In this case $x^* = \sqrt{1.25}$. Numerical experiments are computed to show the bifurcation diagram with respect to k and the Lyapunov numbers . Fig. 1 show the bifurcation diagrams with respect to the parameter k of the orbit of the point (0.1,0.1). In this figure one observes complex dynamic behavior such as cycles of higher order and chaos. Fig. 3 show the Lyapunov numbers of the same orbit for $k = 0.16$. From these results when all parameters are fixed and only k is varied the structure of the game becomes complicated through period doubling bifurcations, more complex bounded attractors are created which are aperiodic cycles of higher order or chaotic attractors.

To demonstrate the sensitivity to initial conditions of the system Eq.(6), we compute two orbits with initial points (0.1, 0.1) and (0.1, 0.1001), respectively. Fig. 2 shows sensitive dependence on initial conditions for y-coordinate of the two orbits, for the system Eq.(6), plotted against the time with the parameters values $a = 12, b = 1, c = 2, k = 0.16$. At the beginning the time series are indistinguishable; but after a number of iterations, the difference between them builds up rapidly. From Fig. 2 we show that the time series of the system Eq. (6) is sensitive dependence to initial conditions, i.e. complex dynamics behaviors occur in this model.

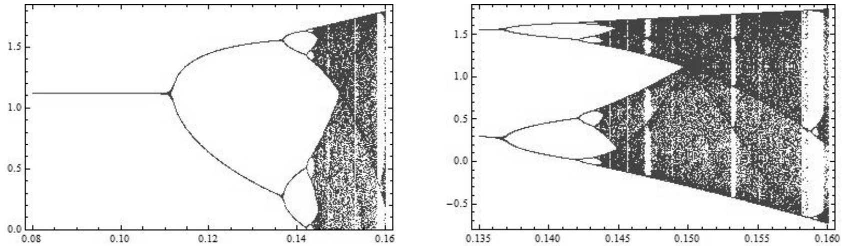

Fig.1. Bifurcation diagrams with respect to the parameter k against variable x or y for $a = 12, b = 1, c = 2$ with 850 iterations of the map Eq. (9).

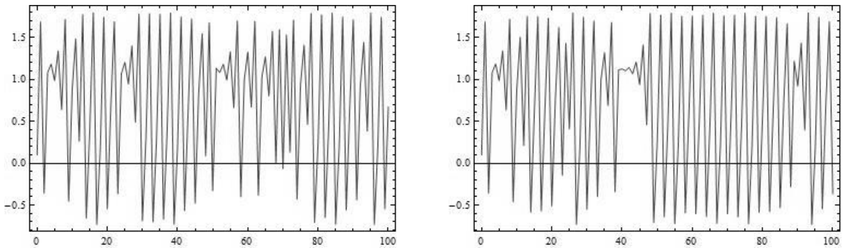

Fig.2. Sensitive dependence on initial conditions, for y–coordinate plotted against the time: The two orbits orb.(0.1, 0.1) (left) and orb.(0.1, 0.1001) (right), for the system (6), with the parameters values a=12,b=1, c = 2, k= 0.16

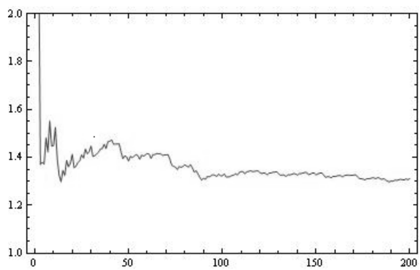

Fig.3. Lyapunov numbers versus the number of iterations of the orbit of the point (0.1, 0.1), for a=12, b=1, c = 2, k= 0.16

Conclusions

In this paper, we analyzed through a discrete dynamical system based on the marginal profits of the players, the dynamics of a nonlinear discrete-time duopoly game, where the players have homogeneous expectations. We suppose that the cost function is linear and the demand function is quadratic. The stability of equilibria, bifurcation and chaotic behavior are investigated. We show that a parameter (speed of adjustment) may change the stability of equilibrium and cause a structure to behave chaotically. For low values of this parameter there is a stable Nash equilibrium. Increasing these values, the equilibrium becomes unstable, through period-doubling bifurcation.

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